Finite Biorthogonal Transforms and Multiresolution Analyses on Intervals

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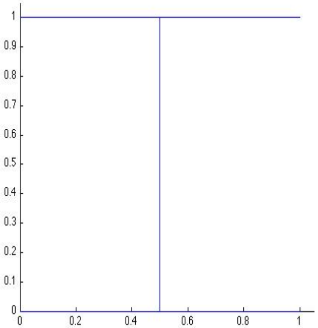
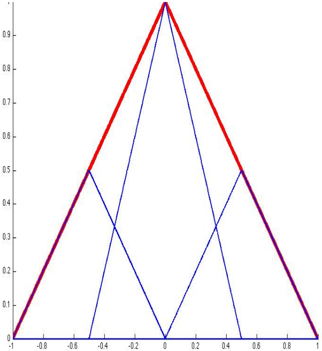
Examples of Scaling Functions

$$\phi(x) = \sqrt{2} \sum_k s_k \phi(2x - k)$$

The characteristic (Haar) function on $[0, 1]$ with coefficients $\{s_0, s_1\} = \frac{1}{\sqrt{2}}\{1, 1\}$

The tent function on $[-1, 1]$ with coefficients $\{s_{-1}, s_0, s_1\} = \frac{1}{\sqrt{2}}\{\frac{1}{2}, 1, \frac{1}{2}\}$

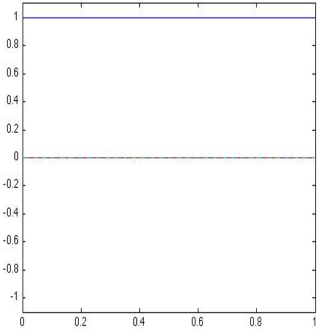


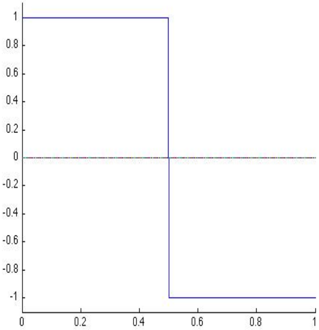


Haar Scaling and Wavelet Functions

Wavelets capture the "finer details."

$$\{s_0, s_1\} = \frac{1}{\sqrt{2}}[1, 1], \quad \{w_0, w_1\} = \frac{1}{\sqrt{2}}[1, -1]$$

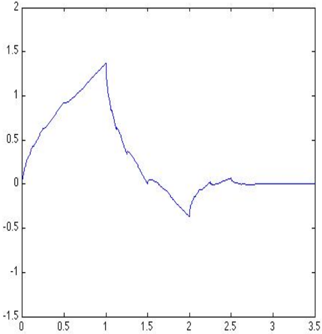


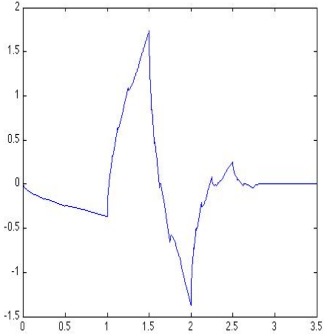


Daubechies Scaling and Wavelet Functions

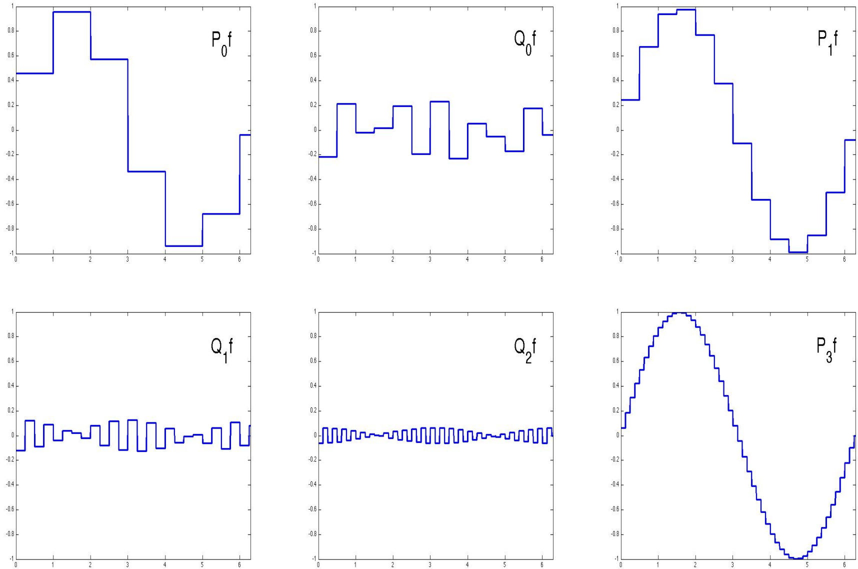
$$\{s_0, s_1, s_2, s_3\} = \frac{1}{\sqrt{2}}\left[\frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\right]$$

$$\{w_0, w_1, w_2, w_3\} = \frac{1}{\sqrt{2}}\left[\frac{1-\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{-1-\sqrt{3}}{4}\right]$$





Projections of a Sine Wave



The Implied MRA from S^∞

$$V_0 = \{\phi_k(x) = \phi(x-k) : k \in \mathbb{Z}\}.$$

The scaling relation gives:

$$\phi\left(\frac{x}{2} - k\right) = \sqrt{2}\sum_j s_j \phi(x - (j+2k)) = \sqrt{2}\sum_m s_{m-2k}\phi(x-m)$$

$$\begin{pmatrix} \vdots \\ \phi_0(\frac{x}{2}) \\ \phi_1(\frac{x}{2}) \\ \phi_2(\frac{x}{2}) \\ \vdots \end{pmatrix} = \sqrt{2} \begin{pmatrix} & \ddots & & & & & & \\ \dots & s_0 & s_1 & s_2 & s_3 & 0 & 0 & \dots \\ \dots & 0 & 0 & s_0 & s_1 & s_2 & s_3 & \dots \\ \dots & 0 & 0 & 0 & 0 & s_0 & s_1 & \dots \\ & & & & \ddots & & & \end{pmatrix} \begin{pmatrix} \vdots \\ \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \vdots \end{pmatrix}$$

Define

$$V_{-1} = \left\{\phi_k\left(\frac{x}{2}\right)\right\} = \left\{\phi\left(\frac{x}{2} - k\right) : k \in \mathbb{Z}\right\}$$