FINITE BIORTHOGONAL TRANSFORMS AND MULTIRESOLUTION ANALYSES ON INTERVALS

DAVID FERRONE

1. Introduction

Wavelet theory over the entire real line is well understood and elegantly presented in various textbooks (e.g. [2], [3], [5]). However the construction of wavelets over finite intervals, which is necessary for many practical applications, does not have one standard definition (it usually depends on the specific application). A typical construction of a multiresolution analysis begins with the intention of using it to encode a string of data (the decomposition of a sequence) into two components. The classical way to guarantee exact, fast reconstruction of this signal from the two components is to use an orthogonal transformation. Another more general method is to use two separate filters - one for decomposition and one for reconstruction. This would involve a biorthogonal transformation (details are available in [1]).

Interpreting [4] as a natural description of orthogonal transformations over finite intervals, we adapt those methods for use in the biorthogonal case. Constructions of multiresolution analyses defined over finite intervals using biorthogonal functions are the focus of this research. A few different methods will be described, as well as regularity and approximation properties.

2. Background

We describe here the orthogonal case. An efficient way to describe the encoding process is via matrix equations. Given scaling coefficients $\{s_0, s_1, \ldots, s_{2n+1}\}$ satisfying

$$\sum_k s_k s_{k-2j} = \delta_{0j} \tag{1}$$

$$\sum_k s_k = 1$$

and defining the wavelet coefficients $w_k = (-1)^k s_{2n+1-k}$, we have two infinite matrices S and W which are orthogonal (i.e. the inverse of S is simply S^*, the complex-conjugate transpose of S) defined by

$$S_{jk} = s_{k-2j}$$
$$W_{jk} = w_{k-2j}$$

Now given a square-summable sequence $f \in \ell^2$ we define its low and high frequency components as

$$\ell = Sf,$$
$$h = Wf.$$

If we are given a sequence $f = \{f_k\}_{k=0}^{2N-1}$ of finite (even) length, we can extend it by periodization, and then apply the previous theory. S and W are then finite; Each are $N \times 2N$ matrices. For example, with $2N = 10$ and $n = 1$ (four scaling coefficients) we would have

$$S = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_0 & s_1 & s_2 & s_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_0 & s_1 & s_2 & s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_0 & s_1 & s_2 & s_3 \\ s_2 & s_3 & 0 & 0 & 0 & 0 & 0 & 0 & s_0 & s_1 \end{pmatrix}.$$

This matrix is almost block-diagonal, but not quite, because of the block in the bottom left corner. The computations involved would be faster if S and W were truly banded matrices. Another issue is that the periodization we introduced has intertwined the data, and decreased the correlation between the encoded signal and the original. That is, the entries at the end of the low and high frequency components ℓ and h come from both the beginning and end of the original signal f.

The goal of [4] was to show that there exist modifications of S and W which are orthogonal and banded. Under some easily verified conditions the coefficients in these modifications can also be used to construct multiresolution analyses over the interval $[0, 2N]$. Regularity of the resulting wavelet functions and their polynomial reproducing properties were explored.

3. The Problem

We now discuss the biorthogonal analogue of the previous setup. Assume that we are given two compactly supported scaling functions

$$\phi = \sum_{k=0}^{2n_s+1} s_k \phi(2x - k),$$

$$\tilde{\phi} = \sum_{k=0}^{2\tilde{n}_s+1} \tilde{s}_k \tilde{\phi}(2x - k),$$

which are biorthogonal

$$\left\langle \phi(x-k), \tilde{\phi}(x-j) \right\rangle_{L^2} = \delta_{jk}$$

and whose integer shifts form Riesz bases. Biorthogonality implies that their scaling coefficients satisfy the *biorthogonality conditions*

$$\sum_k s_k \tilde{s}_{k-2j} = \sum_k w_k \tilde{w}_{k-2j} = 2\delta_{0j}$$
$$\sum_k s_k \tilde{w}_{k-2j} = \sum_k \tilde{s}_k w_{k-2j} = 0$$

for all $j, k \in \mathbb{Z}$ where $w_k = (-1)^k \tilde{s}_{2n+1-k}$, $\tilde{w}_k = (-1)^k s_{2n+1-k}$ (here $n = \max(n_s, \tilde{n}_s)$). We assume there are an even number of coefficients, setting the last equal to zero if necessary. We can then produce the matrices (in the same manner as the orthogonal case) S $\widetilde{S}$, W, and $\widetilde{W}$, and notice that the biorthogonality conditions imply that these matrices satisfy

$$S\widetilde{S}^* = W\widetilde{W}^* = 2I.$$

The first goal is to modify these matrices so that they still satisfy this equation, but are also banded. We can then see in what ways the coefficients of the modifications can be used to form (dual) multiresolution analyses, and discuss the regularity and approximation properties of the resulting functions.

4. Results

We sketch here a basic solution to the problem in the last paragraph, in the case of $n = 2$ (6 scaling coefficients) for simplicity. (The proofs necessary for this case are sufficiently general to demonstrate the result for any n.) The proof of the existence of modifications in [4] (the orthogonal case) relies on the observation that (1) implies

$$S_a = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 \\ 0 & 0 & s_0 & s_1 \end{pmatrix}$$

is orthogonal to

$$S_b = \begin{pmatrix} s_4 & s_5 & 0 & 0 \\ s_2 & s_3 & s_4 & s_5 \end{pmatrix}.$$

Or in other terms,

$$S_a S_b^* = S_b S_a^* = 0_{n,n}. \tag{2}$$

For $2N = 10$ the original orthogonal matrix S is

$$S = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 \\ s_4 & s_5 & 0 & 0 & 0 & 0 & s_0 & s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 & s_5 & 0 & 0 & 0 & 0 & s_0 & s_1 \end{pmatrix}$$

and we notice that the problematic block in the bottom-left of S is S_b. Let M be a modification of S,

$$M = SV,$$

where V is defined so as to remove the S_b block. Explicitly, V is the $2N \times 2N$ matrix

$$V = \begin{pmatrix} (R_a S_a)^* & 0 & (R_b S_b)^* \\ 0 & I_{2N-2n,2N-2n} & 0 \end{pmatrix},$$

where

$$R_a = (S_a S_a^*)^{-1/2} \quad R_b = (S_b S_b^*)^{-1/2}.$$

It is evident that R_a and R_b are well-defined because of the linear independence of the rows of S_a and S_b, and that (2) implies this is the correct choice for V.

Let $N = WV$ be a banded (because of (2)) modification of the wavelet coefficient matrix. The proof that the coefficients in the modified matrices M and N can be used to define a collection of orthogonal scaling and wavelet functions is somewhat technical, but follows from (1) and L^2-convergence of the cascade algorithm, that is, fixed point iteration of the functional equation

$$\phi(x) = \sum_k s_k \phi(2x - k).$$

This L^2 convergence is a result of Cohen's condition, a well-known necessary condition for scaling coefficients satisfying the hypotheses to produce an orthogonal scaling function. (See [2] for details.)

The biorthogonal case can be treated in a similar manner. Defining ϕ and $\tilde{\phi}$ as in section 3, we assume that $\tilde{n}_s \leq n_s =: n$, and that $N > 2n_s$. In the case of $n = 2$ and $2N = 10$ we have the matrix S shown above. We also have $\widetilde{S}$, with the same dimensions and structure as S (though more of the entries in $\widetilde{S}$ may be zero, because it is defined using the dual coefficients $\{\tilde{s}_k\}$). We also need to make use of the wavelet coefficients and matrices, W and $\widetilde{W}$.

Recall that the biorthogonality conditions in section 3 tell us $S\widetilde{S}^* = W\widetilde{W}^* = 2I$, and we wish to preserve this property in our banded modifications M, $\widetilde{M}$, N, and $\widetilde{N}$. This time the key observation is that, using definitions analogous to the orthogonal case, the biorthogonality conditions imply

$$S_a\widetilde{S}_b^* = S_b\widetilde{S}_a^* = \widetilde{S}_a S_b^* = \widetilde{S}_b S_a^* = 0,$$

$$S_a\widetilde{W}_b^* = S_b\widetilde{W}_a^* = \widetilde{S}_a W_b^* = \widetilde{S}_b W_a^* = 0.$$

We then define V so as to remove the $\widetilde{S}_b$ block from the bottom-left corner of $\widetilde{S}$, and $\widetilde{V}$ so that it removes the S_b block from the bottom-left corner of S. We let

$$V = \begin{pmatrix} (R_a S_a)^* & 0 & (R_b S_b)^* \\ 0 & I_{2N-2n_s,2N-2n_s} & 0 \end{pmatrix}$$

$$\widetilde{V} = \begin{pmatrix} \widetilde{W}_a^* & 0 & \widetilde{W}_b^* \\ 0 & I_{2N-2n_s,2N-2n_s} & 0 \end{pmatrix}$$

where R_a and R_b are defined so that $V\widetilde{V}^* = I$. That is,

$$R_a = (S_a\widetilde{W}_a^*)^{-1}, \quad R_b = (S_b\widetilde{W}_b^*)^{-1}.$$

We can then define

$$M = S\widetilde{V}, \quad \widetilde{M} = \widetilde{S}V$$

and confirm that these are banded and satisfy $M\widetilde{M}^* = 2I$. The same statements hold for $N = W\widetilde{V}$, $\widetilde{N} = \widetilde{W}V$. The proof that primal and dual scaling and wavelet functions are well-defined and biorthogonal using these modified coefficients is almost the same as the orthogonal case, making use of the biorthogonality conditions and an L^2-convergence result for the cascade algorithm applied to biorthogonal scaling functions. This too is a necessary condition for ϕ and $\tilde{\phi}$ to be biorthogonal. It is effectively Cohen's condition in biorthogonal form, with details in [1].

5. Present and Future Work

One issue that has not yet been addressed is the domain of the biorthogonal transform described above. For some scaling coefficients the matrix $S_a\widetilde{W}_a^*$ used to define R_a will be singular. So far an example of this which also produces a Riesz basis, and is dual to another scaling function has not been found, and it's plausible that no such example exists for technical reasons. Nevertheless many interesting and practical scaling coefficients are in the domain of this transform so it is worth consideration.

Regardless of the domain of the transform above, it is of interest to generalize the method of its construction. Non-square transforms can be produced that may give similar results provided one is willing to do some pre or post-processing on the signal. Such transforms could be constructed specifically to work with particular scaling sequences (for example, scaling sequences for which the transform above fails).

In the future we hope to generalize this theory in the following ways:

- To allow for various extension techniques besides periodization.
- To find different techniques, or different proofs, for construction of fast finite transforms based on scaling coefficients.
- To use these modifications to describe other function spaces (besides $L^2([0, 2N])$) via MRA over compact domains.

References

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